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# SOLUTION OF HEAT-CONDUCTION PROBLEMS WITH VARIABLE THERMOPHYSICAL PROPERTIES BY THE METHOD OF NETS

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The paper presents the numerical solution of a two-dimensional heat-conduction problem for an infinite bar with a square cross-section, temperature-dependent thermophysical properties and radiant heat transfer at the surface.

Consider the problem of the heating of an infinite bar with a square cross-section A × A with radiant heat transfer at the surface according to the Stefan-Boltzmann law and with temperature-dependent thermophysical properties.

The differential equation of heat conduction is

$$c(T)\,\gamma\,\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left[\lambda(T)\frac{\partial T}{\partial x}\right]+\frac{\partial}{\partial y}\left[\lambda(T)\frac{\partial T}{\partial y}\right]. \tag{1}$$

The boundary conditions are

$$\left.\begin{matrix} x=0, A \\ y=0, A \end{matrix}\right\}\quad \left|\pm\lambda(T)\frac{\partial T}{\partial n}\right|_{s}=\sigma_a(T_0^4-T_s^4), \tag{2}$$

when the origin of the coordinate system (x = y = 0) lies on one of the edges of the bar. At $\tau = 0$

$$T(x,\ y,\ 0)=T_i. \tag{3}$$

In view of the nonlinearity of the differential equation and of boundary conditions (2) the problem will be solved numerically, by the method of nets.

Rewrite equation (1) in the form

$$c\,\gamma\,\frac{\partial T}{\partial \tau}=\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{\partial\lambda}{\partial x}\,\frac{\partial T}{\partial x}+\frac{\partial\lambda}{\partial y}\,\frac{\partial T}{\partial y}. \tag{4}$$

Let us divide the total time of the process into small intervals $\Delta\tau$ (not necessarily equal) and let us replace the derivative $\partial T/\partial\tau$ by the expression $(T_{m,k,\tau+\Delta\tau}-T_{m,k,\tau})/\Delta\tau$. In addition to that, let us express the coefficients $\lambda$ and c in terms of the temperatures at the nodes of a two-dimensional space grid at the beginning of each time interval $\Delta\tau$, and let us introduce the finite-difference approximations

$$\frac{\partial T}{\partial x}=\frac{T_{m+1,k,\tau}-T_{m-1,k,\tau}}{2\Delta x},\quad \frac{\partial\lambda}{\partial x}=\frac{\lambda_{m+1,k,\tau}-\lambda_{m-1,k,\tau}}{2\Delta x},$$

$$\frac{\partial^2 T}{\partial x^2}=\frac{T_{m+1,k,\tau}-2T_{m,k,\tau}+T_{m-1,k,\tau}}{\Delta x^2}\quad \text{etc.},$$

This yields a finite-difference equation of the explicit type

$$T_{m,k,\tau+\Delta\tau}=T_{m,k,\tau}+\frac{a_{m,k,\tau}\Delta\tau}{\Delta x^2}\left[\left(1+\frac{\lambda_{m+1,k,\tau}-\lambda_{m-1,k,\tau}}{4\lambda_{m,k,\tau}}\right)\times\right.$$

$$\times(T_{m+1,k,\tau}-T_{m,k,\tau})+\left(1-\frac{\lambda_{m+1,k,\tau}-\lambda_{m-1,k,\tau}}{4\lambda_{m,k,\tau}}\right)\times \tag{5}$$

$$\left.\times(T_{m-1,k,\tau}-T_{m,k,\tau})\right]+\frac{a_{m,k,\tau}\Delta\tau}{\Delta y^2}\left[\left(1+\frac{\lambda_{m,k+1,\tau}-\lambda_{m,k-1,\tau}}{4\lambda_{m,k,\tau}}\right)\times\right.$$

$$\times (T_{m,\,k+1,\,\tau} - T_{m,\,k,\,\tau}) + \left(1 - \frac{\lambda_{m,\,k+1,\,\tau} - \lambda_{m,\,k-1,\,\tau}}{4\lambda_{m,\,k,\,\tau}}\right) \times$$

$$\times (T_{m,\,k-1,\,\tau} - T_{m,\,k,\,\tau})\Big] . \quad (5) \text{ (cont'd)}$$

The choice of the time interval $\Delta\tau$ is determined by the stability condition of equation (5).

The boundary conditions (2) are introduced in the following manner [1]: Let the net cover the cross-section of the body in such a way that the boundaries of the cross-section lie midway between the two outer rows on each side (Fig. 1). Introducing the approximations

$$\frac{\partial T}{\partial n} = \frac{T_{\mathrm{f}\,k,\,\tau} - T_{1,\,k,\,\tau}}{h}, \quad \frac{\partial T}{\partial n} = \frac{T_{m,\,\mathrm{f},\,\tau} - T_{m,\,1,\,\tau}}{h}, \quad (6)$$

$$T_{\mathrm{s},\,k,\,\tau} = \frac{1}{2}(T_{1,\,k,\,\tau} + T_{\mathrm{f},\,k,\,\tau}), \quad T_{m,\,\mathrm{s},\,\tau} = \frac{1}{2}(T_{m,\,1,\,\tau} + T_{m,\,\mathrm{f},\,\tau}), \quad (7)$$

into (2), we obtain the expression for the temperature at the outer (fictitious) point of the boundary semi-layer

$$T_{\mathrm{f},\,k,\,\tau} = \frac{1}{1 + \alpha h/2\lambda}\left[\left(1 - \frac{\alpha h}{2\lambda}\right) T_{1,\,k,\,\tau} + \frac{\alpha h}{\lambda} T_{\mathrm{c},\,\tau}\right], \quad (8)$$

where

$$\alpha = \sigma_{\mathrm{a}}(T^4_{\mathrm{O},\,\tau} - T^4_{\mathrm{s},\,k,\,\tau})/(T_{\mathrm{O},\,\tau} - T_{\mathrm{s},\,k,\,\tau}). \quad (9)$$

In view of the fact that the heat-transfer coefficient $\alpha$ is a function of the unknown temperature at the surface, boundary condition (2) can be introduced by successive approximations, iterating with respect to either $T_{\mathrm{f},\,\tau}$ or $T_{\mathrm{s},\,\tau}$. In order to

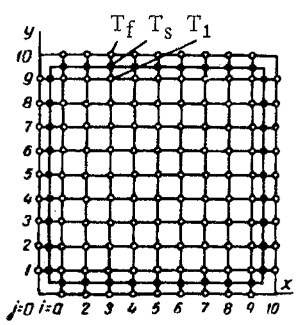


Fig. 1. Net of points for the calculation of the heating of a solid body with a square cross-section.

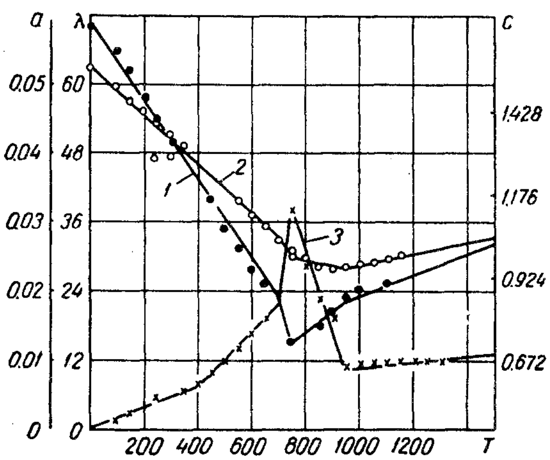


Fig. 2. Dependence of the thermophysical properties of Steel 08 on temperature, according to [2]. 1) a ($m^2$/hr); 2) $\lambda$ (W/m · deg); 3 c (kJ/kg · deg).

speed up the convergence of successive approximations we can use Newton's formula

$$T^{(\rho+1)}_{\mathrm{f},\,k,\,\tau} = T^{(\rho)}_{\mathrm{f},\,k,\,\tau} - f[T^{(\rho)}_{\mathrm{f},\,k,\,\tau}]/f'[T^{(\rho)}_{\mathrm{f},\,k,\,\tau}], \quad (10)$$

where $\rho$ is the ordinal number of the iteration.

The function $f(T_{\mathrm{f}})$ is obtained by transferring all terms of equation (2) to the right-hand side:

$$f(T_{\mathrm{f}}) = \lambda\frac{\partial T}{\partial n} - \sigma_{\mathrm{a}}(T^4_{\mathrm{O},\,\tau} - T^4_{\mathrm{s},\,k,\,\tau}).$$

Taking account of (6) and (7) we obtain

$$f(T_f) = \frac{\lambda}{h}(T_{f,\,k,\,\tau} - T_{1,\,k,\,\tau}) - \sigma_a\left[T_{0,\,\tau}^4 - \frac{1}{16}(T_{f,\,k,\,\tau} + T_{1,\,k,\,\tau})^4\right], \tag{11}$$

$$f'(T_f) = \frac{\lambda}{h} + \frac{\sigma_a}{4}(T_{f,\,k,\,\tau} + T_{1,\,k,\,\tau})^3 \ldots\ldots \tag{12}$$

and substituting these into (10) yields

$$T_{f,\,k,\,\tau}^{(p+1)} = T_{f,\,k,\,\tau}^{(p)} - \left\{\frac{\lambda}{h}\left(T_{f,\,k,\,\tau}^{(p)} - T_{1,\,k,\,\tau}\right) - \sigma_a\left[T_{0,\,\tau}^4 - \frac{1}{16}\left(T_{f,\,k,\,\tau}^{(p)} + T_{1,\,k,\,\tau}\right)^4\right]\right\} \times$$
$$\times\left[\frac{\lambda}{h} + \frac{\sigma_a}{4}\left(T_{f,\,k,\,\tau}^{(p)} + T_{1,\,k,\,\tau}\right)^3\right]^{-1}. \tag{13}$$

As the initial approximation $T_{f,\,k,\tau}^{(0)}$ we can take the value of $T_f$ obtained from equations (8) and (9) with the surface temperature corresponding to the beginning of the time interval $\Delta\tau$.

The results show that the successive approximations based on Newton's formula converge very rapidly. The second and third approximations already agree to four significant figures.

Values of Thermal Conductivity λ and Thermal Diffusivity $a$ of Steel 08 According to [2]
($k_{\lambda n}$ and $k_{an}$—temperature coefficients in equations (15))

| $T_n$, °K | $\lambda_n$, W/m·deg | $a_n$, m²/hr | $k_{\lambda n}$, 1/deg | $k_{an}$, 1/deg |
|---|---|---|---|---|
| 273 | 61.44 | 0.059 | $-0.635\cdot10^{-3}$ | $-0.974\cdot10^{-3}$ |
| 673 | 45.84 | 0.036 | $-0.907\cdot10^{-3}$ | $-0.157\cdot10^{-2}$ |
| 973 | 33.36 | 0.019 | $-0.216\cdot10^{-2}$ | $-0.737\cdot10^{-2}$ |
| 1023 | 29.76 | 0.012 | $-0.325\cdot10^{-3}$ | $+0.250\cdot10^{-2}$ |
| 1223 | 27.84 | 0.018 | $+0.376\cdot10^{-3}$ | $+0.100\cdot10^{-2}$ |
| 1773 | 33.60 | 0.027 | — | — |

Note: The values of the thermophysical properties in the temperature range 1223 to 1773° K were obtained by extrapolation of the experimental data of [5].

As an illustration, we shall show the results of the computation of the radiant heating of a square bar. The data are typical for the heating of ingots with square cross-section in a soaking pit before rolling. The heating is assumed to be in two stages: a) raising of the temperature of the heating medium according to a linear law

$$T_0 \equiv T_{furn} = T_{furn,0} + b\tau, \tag{14}$$

with $T_{furn,\,0}$ = 973°K and b = 175 deg/hr; b) heating at constant furnace temperature $T_{furn}$ = 1673°K.

The initial temperature of the ingot is assumed to be 293°K. The apparent coefficient of radiant heat transfer is $\sigma_a = 3.5\cdot10^{-8}$ W/m² deg⁴.

The variation of the thermophysical properties of Mark 08 steel [2] was approximated, as shown in Fig. 2, by piecewise linear functions

$$\lambda_{m,\,k} = \lambda_{n-1}[1 + k_{\lambda n}(T_{m,\,k} - T_{n-1})],$$
$$a_{m,k} = a_{n-1}[1 + k_{an}(T_{m,\,k} - T_{n-1})]. \tag{15}$$

The values of the coefficients $\lambda_{n-1}$, $a_{n-1}$, $k_{\lambda n}$ and $k_{an}$ as functions of the temperatures $T_{n-1}$ are given in the table. The results of the computation of the heating of ingots with A = 0.4-1.0 m are given in Fig. 3.

As can be seen from the graphs, in the temperature range in the neighborhood of 1050°K the temperature increase at the axis of the ingot slows down, due to the absorption of heat by structural transformations. A similar effect has been observed experimentally [3]. The results shown in Fig. 3 were used to optimize the process of preheating of steel ingots before rolling.

It should be noted that the problems of stability and convergence of finite-difference equations for nonlinear hear-conduction problems have not yet been analyzed in a satisfactory manner.

The finite-difference equation (5) is stable when the time step satisfies the condition [4]

$$(\Delta\tau)_\tau \leqslant (\Delta\tau_{\min})_\tau = Kh^2/(a_{m,\,k,\,\tau})_{\max}\,, \tag{16}$$

where K = 1/4.

In order to check the validity of condition (16), the computation of the heating of a square bar with A = 0.72 m was repeated with different values of K, viz. K = 1/4, 1/8, and 1/12, with all other conditions being equal (the number of space steps in each direction was 10).

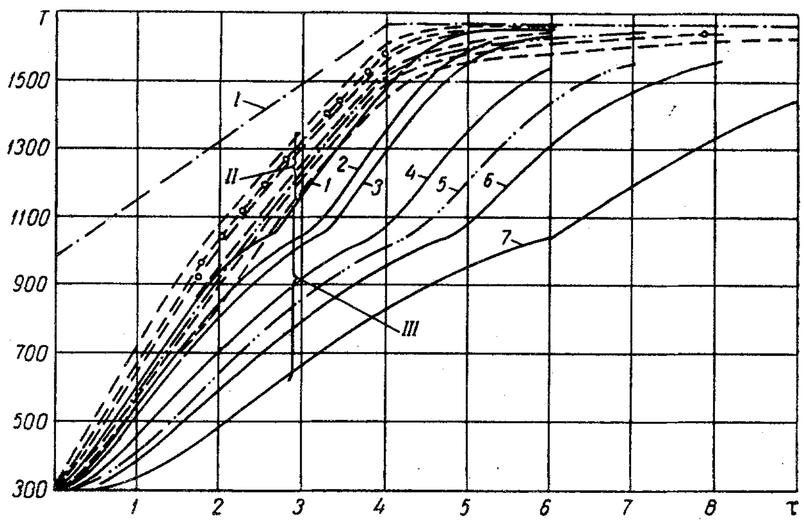


Fig. 3. Results of the computation of the heating of square ingots in a soaking pit (cross-section A = B = 0.4-1.0 m; material – Steel 08; $\sigma_a = 3.5 \cdot 10^{-8}$ W/m$^2$deg$^4$; $T_i$ = 293°K.): I) $T_{furn}$; II) $T_{surface}$; III) $T_{axis}$ for 1) A = 400 mm; 2) 480; 3) 520; 4) 640; 5) 720; 6) 800; 7) 1000.

A comparison of the results shows that the reduction of the value of K = 1/4 by a factor of two or three led to deviations not higher than 3 deg at the axis and not higher than 7 deg at the center of each side at the initial stage of the heating, and these deviations subsequently continuously decreased. This confirms the stability of the computation according to equation (5) under the condition

$$(\Delta\tau)_\tau = h^2/4a_{\max,\,\tau}\,. \tag{16a}$$

To estimate the error associated with the approximation of the differential equation by a finite-difference equation, we used Runge's method [5], based on a comparison of the results of computations with different mesh sizes.

Let $\varepsilon_{N_1}$ and $\varepsilon_{N_2}$ be the errors in the values of the temperature when the cross-section is divided into $N_1$ and $N_2$ layers, respectively, i.e.,

$$T = T_{N1} + \varepsilon_{N1}\,,\ T = T_{N2} + \varepsilon_{N2}\,,\ T_{N1} - T_{N2} = \varepsilon_{N2} - \varepsilon_{N1}\,. \tag{17}$$

We assume that the error is proportional to the square of the step size, i.e., $\varepsilon = k(\Delta x^2 + \Delta y^2)$. When $\Delta x = \Delta y = h$

$$\begin{aligned} \varepsilon_{N1} &= 2kh_{N1}^2\,,\quad \varepsilon_{N2} = 2kh_{N2}^2\,,\\ \varepsilon_{N2} - \varepsilon_{N1} &= 2k\,(h_{N2}^2 - h_{N1}^2). \end{aligned} \tag{18}$$

Comparing (17) with (18), taking into account that $h_{N_2} = (N_1/N_2)\,h_{N_2}$, we obtain

$$\varepsilon_{N1} = \frac{T_{N1} - T_{N2}}{(N_1/N_2)^2 - 1}\,. \tag{19}$$

A comparison of the results for a bar with A = 0.5 m of Steel 08, divided into 7 and 14 layers (in each direction), indicates that the error of the computation with N = 7 is not higher than 10 deg at the axis and at the centers of the sides at the initial stage of the heating, and subsequently (for $\tau > 2$ hr) the error decreases to a few degrees.

Thus the accuracy of the solution obtained with a net of 7-10 steps in each direction, and with a time step satisfying condition (16a), is quite sufficient for practical purposes.

It should be noted that the choice of too small steps h and $\Delta\tau$ results in a sharp increase of machine time. Thus it is interesting to consider net equations for heat-conduction problems which are not restricted by a stability condition for the time step $\Delta\tau$.

Such equations are described in the literature [4]. In particular, there have been developed such equations for two-dimensional problems, using a square space net, but in most cases the thermophysical properties were assumed to be constant.

In the following we shall derive net equations of the explicit and implicit types which are not restricted in the choice of the space and time steps, suitable for problems with variable thermophysical properties.

Substituting in equation (4) the finite-difference approximations for the derivatives

$$\frac{\partial T}{\partial \tau}=\frac{T_{m,\,k,\,\tau}-T_{m,\,k,\,\tau-\Delta\tau}}{\Delta\tau},\quad \frac{\partial T}{\partial x}=\frac{T_{m+1,\,k,\,\tau}-T_{m-1,\,k,\,\tau}}{2\Delta x},$$

$$\frac{\partial \lambda}{\partial x}=\frac{\lambda_{m+1,\,k,\,\tau-\Delta\tau}-\lambda_{m-1,\,k,\,\tau-\Delta\tau}}{2\Delta x},$$

$$\frac{\partial^2 T}{\partial x^2}=\frac{T_{m+1,\,k,\,\tau}-2T_{m,\,k,\,\tau}+T_{m-1,\,k,\,\tau}}{\Delta x^2}\quad \text{etc.},$$

we obtain the implicit net equation

$$AT_{m,\,k,\,\tau}+BT_{m+1,\,k,\,\tau}+CT_{m-1,\,k,\,\tau}+DT_{m,\,k+1,\,\tau}+ ET_{m,\,k-1,\,\tau}=-S_0T_{m,\,k,\,\tau-\Delta\tau}, \tag{20}$$

where

$$A=-\left[2\frac{a_{m,\,k,\,\tau-\Delta\tau}}{a_0}\left(1+\frac{\Delta x^2}{\Delta y^2}\right)+S_0\right],$$

$$B=\frac{a_{m,\,k,\,\tau-\Delta\tau}}{a_0}\left(1+\frac{\lambda_{m+1,\,k,\,\tau-\Delta\tau}-\lambda_{m-1,\,k,\,\tau-\Delta\tau}}{4\lambda_{m,\,k,\,\tau-\Delta\tau}}\right),$$

$$C=\frac{a_{m,\,k,\,\tau-\Delta\tau}}{a_0}\left(1-\frac{\lambda_{m+1,\,k,\,\tau-\Delta\tau}-\lambda_{m-1,\,k,\,\tau-\Delta\tau}}{4\lambda_{m,\,k,\,\tau-\Delta\tau}}\right),$$

$$D=\left(\frac{\Delta x}{\Delta y}\right)^2\frac{a_{m,\,k,\,\tau-\Delta\tau}}{a_0}\left(1+\frac{\lambda_{m,\,k+1,\,\tau-\Delta\tau}-\lambda_{m,\,k-1,\,\tau-\Delta\tau}}{4\lambda_{m,\,k,\,\tau-\Delta\tau}}\right),$$

$$E=\left(\frac{\Delta x}{\Delta y}\right)^2\left(1-\frac{\lambda_{m,\,k+1,\,\tau-\Delta\tau}-\lambda_{m,\,k-1,\,\tau-\Delta\tau}}{4\lambda_{m,\,k,\,\tau-\Delta\tau}}\right)\frac{a_{m,\,k,\,\tau-\Delta\tau}}{a_0},$$

$a_0$ is an arbitrary constant, e.g., $a_0 = a(273°\text{K})$, and $S_0 = \Delta x^2/a_0\Delta\tau$. Equation (20) is a finite-difference equation of the elliptic type, which can be solved either by direct (matrix) or by iterative methods [4].

To obtain an explicit net equation which is stable in a wide range of step sizes $\Delta x$, $\Delta y$, $\Delta\tau$, we shall use the method of Du Fort and Frankel [6]. Approximate the time derivative by the central difference

$$\frac{\partial T}{\partial \tau}=\frac{1}{2\Delta\tau}(T_{m,\,k,\,\tau+\Delta\tau}-T_{m,\,k,\,\tau-\Delta\tau}).$$

Using the relation

$$T_{m,\,k,\,\tau}=\frac{1}{2}(T_{m,\,k,\,\tau+\Delta\tau}+T_{m,\,k,\,\tau-\Delta\tau})$$

and the finite-difference approximations for the space derivatives listed above, we obtain the explicit net equation

$$T_{m,\,k,\,\tau+\Delta\tau} = \frac{\omega - (1 + \Delta x^2/\Delta y^2)\,a_{m,\,k,\,\tau}/a_0}{\omega + (1 + \Delta x^2/\Delta y^2)a_{m,\,k,\,\tau}/a_0}\, T_{m,\,k,\,\tau-\Delta\tau} +$$

$$+ \frac{a_{m,\,k,\,\tau}/a_0}{\omega + (1 + \Delta x^2/\Delta y^2)\,a_{m,\,k,\,\tau}/a_0}\left[\left(1 + \frac{\lambda_{m+1,\,k,\,\tau} - \lambda_{m-1,\,k,\,\tau}}{4\,\lambda_{m,\,k,\,\tau}}\right) T_{m+1,\,k,\,\tau} +\right.$$

$$+ \left(1 - \frac{\lambda_{m+1,\,k,\,\tau} - \lambda_{m-1,\,k,\,\tau}}{4\,\lambda_{m,\,k,\,\tau}}\right) T_{m-1,\,k,\,\tau} + \tag{21}$$

$$+ \frac{\Delta x^2}{\Delta y^2}\left(1 + \frac{\lambda_{m,\,k+1,\,\tau} - \lambda_{m,\,k-1,\,\tau}}{4\,\lambda_{m,\,k,\,\tau}}\right) T_{m,\,k+1,\,\tau} +$$

$$\left. + \frac{\Delta x^2}{\Delta y^2}\left(1 - \frac{\lambda_{m,\,k+1,\,\tau} - \lambda_{m,\,k-1,\,\tau}}{4\,\lambda_{m,\,k,\,\tau}}\right) T_{m,\,k-1,\,\tau}\right],$$

where

$$\omega = \Delta x^2/2a_0\,\Delta\tau, \quad a_0 = \text{const}.$$

In the case of constant $\lambda$ and $a$ and $\Delta x = \Delta y$ equation (21) reduces to the equation of Du Fort and Frankel [6]

$$T_{m,\,k,\,\tau+\Delta\tau} = \frac{\omega - 2}{\omega + 2} T_{m,\,k,\,\tau-\Delta\tau} + \frac{1}{\omega + 2}(T_{m+1,\,k,\,\tau} + T_{m-1,\,k,\,\tau} + T_{m,\,k+1,\,\tau} + T_{m,\,k-1,\,\tau}). \tag{21a}$$

Equation (21) requires that the values of the temperature at all space points be stored for three consecutive time levels ($\tau - \Delta\tau$, $\tau$, $\tau + \Delta\tau$), unlike equation (5) which involves only two levels.

The comparison of equations (5), (20), and (21) from the point of view of applicability to machine computation, as well as the comparison of their stability and convergence, are outside the scope of the present work.

## NOTATION

n – surface normal; $T_0$, $T_{furn}$ – temperature of heating medium and furnace, respectively; $T_i$ – initial temperature of heated solid; $\lambda$, c, $\gamma$, $a$ – thermal conductivity, specific heat, density, and thermal diffusivity of the material; $\tau$ – time; A – length of side of square cross-section; $\Delta x = A_1/N_1$, $\Delta y = A_2/N_2$, h = A/N – width of layers (distance between nodes) in finite-difference net; $\sigma_a$ – apparent coefficient of radiant heat transfer; m = 1, 2, 3, ..., k = 1, 2, 3, ... – ordinal numbers of net nodes along the x and y coordinates, respectively.

## REFERENCES


1. P. P. Yushkov, Trudy Instituta Energetiki AN BSSR, vol. 6, 1958.
2. Thermophysical Properties of Materials [in Russian], handbook ed., N. B. Vargaftik, GEI, 1956.
3. E. P. Blokhin, et al., Tr. VNIIMT, no. 10, 1963.
4. V. K. Saul'ev, Integration of Equations of Parabolic Type by the Method of Nets [in Russian], Fizmatgiz, 1960.
5. D. Yu. Panov, Handbook for the Numerical Solution of Partial Differential Equations [in Russian], Gostekhizdat, 1943.
6. E. C. Du Fort and S. P. Frankel, Math. Tables and Other Aids Comput., 7, 43, (21), 1953.


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